An Introduction to

Mathematical Reasoning

numbers, sets and functions

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The language of mathematics

Pure mathematics is concerned with the exploration of mathematical concepts arising initially from the study of space and number. In order to capture and communicate mathematical ideas we must make statements about mathematical objects and much mathematical activity can be described as the formulation of mathematical statements and then the determination of whether or not such statements are true or false. It is important to be clear about what constitutes a mathematical statement and this is considered in this first chapter. We begin with simple statements and then examine ways of building up more complicated statements.

1.1 Mathematical statements

It is quite difficult to give a precise formulation of what a mathematical statement is and this will not be attempted in this book. The aim here is to enable the reader to recognize simple mathematical statements. First of all let us consider the idea of a *proposition.* A good working criterion is that *a proposition is a sentence which is either true or false (but not both).* For the moment we are not so concerned about whether or not propositions are in fact true. Consider the following list.

(i) $1 + 1 = 2$.
(ii) $\pi = 3$.
(iii) 12 may be written as the sum of two prime numbers.
(iv) Every even integer greater than 2 may be written as the sum of two prime numbers.
(v) The square of every even integer is even.
(vi) n is a prime number.

(vii) $n^2 - 2n > 0$.

(viii) $m < n$.

(ix) $12 - 11$.

(x) π is a special number.

Of these the first five are propositions. This says nothing about whether or not they are true. In fact (i) is true and (ii) is false. Proposition (iii) is true since $12 = 7 + 5$ and 5 and 7 are prime numbers.† Propositions (iv) and (v) are general statements which cannot be proved by this sort of simple arithmetic. In fact it is easy to give a general proof of (v) (see Exercise 3.3) showing that it is true. However, at the time of writing, it is unknown whether (iv) is true or false; this statement is called the Goldbach conjecture after Christian Goldbach who suggested that it might be true in a letter to Leonhard Euler written in 1742.

On the other hand the others on the above list are not propositions. The next two, (vi) and (vii), become propositions once a numerical *value* is assigned to n. For example if $n = 2$ then (vi) is true and (vii) is false, whereas if $n = 3$ then they are both true. The next, (viii), becomes a proposition when values are assigned to both m and n, for example it is true for $m = 2$, $n = 3$ and false for $m = 3$, $n = 2$. Sentences of this type are called *predicates*. The symbols which need to be given values in order to obtain a proposition are called *free variables*.

The word *statement* will be used to denote either a proposition or a predicate. So in the above list the first eight items are statements. We will use a single capital letter P or Q to indicate a statement, or sometimes an expression like $P(m, n)$ to indicate a predicate, with the free variables listed in brackets.

The last two entries in the above list are not statements: (ix) is not even a sentence and (x) doesn't mean anything until we know what 'special' means. Very often mathematicians do give technical meanings to everyday words (as in 'prime' number used in (iii) and (iv) above and 'even' number used in (v)) and so if 'special' had been given such a meaning as a possible property of a number then (x) would be a proposition and so a statement. Of course the fact that (i) to (viii) are statements relies on a number of assumptions about the meanings of the symbols and words which have been taken for granted.

† A positive integer is a *prime number* if it is greater than 1 and is divisible only by 1 and itself. Thus the first few prime numbers are 2, 3, 5, 7, 11, ... This definition is discussed in Chapter 23.

In particular it is quite complicated to give a precise definition of the number π in statement (ii). The number π was originally defined geometrically as the ratio between the length of the circumference of a circle and the length of the diameter of the circle. In order for this definition to be justified it is necessary to define the length of the circumference, a curved line, and having done that to prove that the ratio is independent of the size of the circle. This was achieved by the Greeks in classical times. A regular hexagon inscribed in a circle of unit diameter has circumference 3 and so it is clear that $\pi > 3$.† Archimedes showed by geometrical means in his book *On the measurement of the circle* that $3\frac{10}{71} < \pi < 3\frac{1}{7}$ and also provided a beautiful proof of the formula πr^2 for the area of a circle of radius r. The modern definition of π is usually as twice the least positive number for which the cosine function vanishes, and the details may be found in any text on analysis or advanced calculus.‡

1.2 Logical connectives

In mathematics we are often faced with deciding whether some given proposition is true or whether it is false. Many statements are really quite complicated and are built up out of simpler statements using various 'logical connectives'. For the moment we restrict ourselves to the simplest of these: 'or', 'and' and 'not'. The truth or falsehood of a complicated statement is determined by the truth or falsehood of its component statements. It is important to be clear how this is done.

The connective 'or'

Suppose that we say

For integers a and b, $ab = 0$ if $a = 0$ or $b = 0$.

The statement '$a = 0$ or $b = 0$' is true if $a = 0$ (regardless of the value of b) and is also true if $b = 0$ (regardless of the value of a). Notice that the statement is true if both $a = 0$ and $b = 0$ are true. This is called

† Some fundamentalist Christians have claimed that 1 Kings vii 23, in which a round object of diameter 10 cubits is stated to have a circumference of 30 cubits, 'proves' that $\pi = 3$ but mathematical truth can never be a matter simply of faith (nor can religious truth either in the author's experience) and in any case there is no suggestion in the biblical passage that the measurements were highly accurate!

‡ See for example R. Haggerty, *Fundamentals of mathematical analysis,* Addison-Wesley, Second edition 1993.

the 'inclusive' use of 'or'. Its meaning is best made precise by means of a *truth table.* Any statement may be either true or false: we say that 'true' and 'false' are the two possible *truth values* for the statement. So given two statements P and Q each has two possible truth values giving four possible combinations in all. The truth table for 'or' which follows specifies the truth value for 'P or Q' corresponding to each possible combination of truth values for P and for Q, one line for each. In the table T indicates 'true' and F indicates 'false'.

Table 1.2.1

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

In this table, the truth values in the second line for example indicate that if P is true and Q is false then the statement 'P or Q' is true.

'P or Q' is called the *disjunction* of the two statements P and Q.

In everyday speech 'or' is often used in the exclusive sense as in the first sentence of this section: 'we are faced with deciding whether some given proposition is true or whether it is false', in which it is implicitly understood (and emphasized by the uses of the word 'whether') that it is not possible for a proposition to be both true and false.

To give another everyday example of the 'exclusive or', when we say 'everyone will travel there by bus or by train' this would normally be taken to mean that everyone uses one or other form of transport but not both. If we wanted to allow for someone using both we would probably say 'everyone will travel there by bus or by train or by both'. In practice the precise meaning of 'or' is made clear by the context or there is some ambiguity (but not both!). The meaning of mathematical statements must be precise and so we avoid these ambiguities by always using 'or' in the inclusive way determined by the above truth table.

The connective 'or' is sometimes hidden in other notation. For example when we write '$a \leqslant b$' where a and b are real numbers this is a shorthand for '$a < b$ or $a = b$'. Thus both the statements '$1 \leqslant 2$' and '$2 \leqslant 2$' are true.

Similarly '$a = \pm b$' is a shorthand for '$a = b$ or $a = -b$'. There is some possibility of confusion here for the true statement that $1 = \pm 1$ does not mean that 1 and ± 1 are interchangeable: 'if $x^2 = 1$ then $x = \pm 1$' is a true statement whereas 'if $x^2 = 1$ then $x = 1$' is a false statement! It is necessary to be clear that $a = \pm b$ is not asserting that a single equality is true but is asserting that one (or both) of two equalities $a = b$, $a = -b$ is true. The following truth table illustrating this may help.

a	b	$a = b$	$a = -b$	$a = \pm b$
1	2	F	F	F
1	1	T	F	T
2	-2	F	T	T
0	0	T	T	T

Observe how the truth values in the final column may be read off from the truth values in the previous two columns using the truth table for 'or' (Table 1.2.1).

In fact the statement $a = \pm b$ can be written as a single equation, namely $|a| = |b|$ where $|a|$ denotes the *absolute value* of a (i.e. $|a| = a$ if $a \geqslant 0$ and $|a| = -a$ if $a \leqslant 0$).

The connective 'and'

We use this when we wish to assert that two things are both true. Again this word can be hidden in other words or notation. Thus if we assert that

π lies between 3 and 4

or in symbols

$3 < \pi < 4$

then this means that

$\pi > 3$ and $\pi < 4$

which is called the *conjunction* of the two statements '$\pi > 3$' and '$\pi < 4$'.

You should construct a truth table for 'and'.

The connective 'not'

Finally we have the idea of the *negation* of a statement. The negation of a statement is true when the original statement is false and it is false when the original statement is true. This is described by the following truth table.

Table 1.2.2

P	not P
T	F
F	T

The negation of a statement is usually obtained by including the word 'not' in the statement but does require a little care since this is often used very loosely in everyday speach.

Example 1.2.3 Consider the following statement about a polynomial $f(x)$ with real coefficients, such as $x^2 + 3$ or $x^3 - x^2 - x$.

(i) For real numbers a, if $f(a) = 0$ then a is positive (i.e. $a > 0$).

What is the negation of this statement? The following possibilities spring to mind from the everyday use of 'negation'.

(ii) For real numbers a, if $f(a) = 0$ then a is negative.
(iii) For real numbers a, if $f(a) = 0$ then a is non-positive.
(iv) For real numbers a, if $f(a) \neq 0$ then a is positive.
(v) For real numbers a, if a is positive then $f(a) = 0$.
(vi) For real numbers a, if a is positive then $f(a) \neq 0$.
(vii) For real numbers a, if a is non-positive then $f(a) \neq 0$.
(viii) For some non-positive real number a, $f(a) = 0$.

If you think about it you will see that none but the last is the negation. For example, if $f(x) = x^3 - x = x(x+1)(x-1)$ then statement (i) is certainly false since, for the real number 0, $f(0) = 0$ but 0 is not positive. On the other hand so are all the other statements apart from (viii) which is true since 0 is a non-positive number such that $f(0) = 0$. We will consider the negation of statements of this form more formally in the next chapter.

Exercises

1.1 Construct a truth table for 'and' as follows.

P	Q	P and Q
T	T	
T	F	
F	T	
F	F	

1.2 Construct truth tables for the statements

(i) not (P and Q);
(ii) (not P) or (not Q);
(iii) P and (not Q);
(iv) (not P) or Q.

1.3 Using the truth table for 'or' complete the following truth table for the statement $a \leqslant b$.

a	b	$a < b$	$a = b$	$a \leqslant b$
1	1			
2	1			
1	2			
−2	1			

1.4 Consider the following statement.

(i) All girls are good at mathematics.

Which of the following statements is the negation of the above statement?

(ii) All girls are bad at mathematics.
(iii) All girls are not good at mathematics.
(iv) Some girl is bad at mathematics.
(v) Some girl is not good at mathematics.
(vi) All children who are good at mathematics are girls.
(vii) All children who are not good at mathematics are boys.

Can you find any statements in this list which have the same meaning as the original statement (i)?

1.5 Prove that $|a|^2 = a^2$ for every real number a.